A demonstration of Alfvén waves

Part 1. Generation of standing waves

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In this paper the equations for Alfvén waves are examined and conditions necessary for the occurrence of resonance are determined. An experiment using liquid sodium inside a torus is described. Waves were generated by means of alternating current supplied to a coil around the torus. A strong resonance was observed at the fundamental frequency, and a weak resonance at three times this frequency. At the fundamental resonance the alternating magnetic field attained magnitudes more than 9 times the magnitude of the field that would have been generated in free space by the exciting current. The results were in good agreement with prediction.

1. Introduction

Previous experiments to demonstrate Alfvén waves in liquids (Lundquist 1949; Lehnert 1954) have produced rather limited effects because of heavy damping. Solutions of the equations for Alfvén waves, however, indicate that it should be possible to produce strong resonant effects in apparatus of manageable size, provided that the fluid region has the right shape. A further experiment to demonstrate these waves in sodium has therefore been undertaken at the Cambridge University Engineering Department.

2. Theory

Neglecting the electrostatic force $q\mathbf{E}$, the convection current $q\mathbf{u}$, and the displacement current $\epsilon(\partial\mathbf{E}/\partial t)$, the equations for an incompressible conducting fluid of electric diffusivity $\lambda = 1/\mu\sigma$ and viscous diffusivity ν are, in the usual notation, with m.k.s. units,

$$\left.\begin{aligned} \nabla\times\mathbf{E} &= -\partial\mathbf{B}/\partial t, \quad \nabla\times\mathbf{B} = \mu\mathbf{j}, \\ \nabla.\mathbf{B} &= 0, \quad \mu\lambda\mathbf{j} = \mathbf{E}+\mathbf{u}\times\mathbf{B}, \\ \partial\mathbf{u}/\partial t+\mathbf{u}.\nabla\mathbf{u} &= \rho^{-1}\mathbf{j}\times\mathbf{B}-\rho^{-1}\nabla p+\nu\nabla^2\mathbf{u}, \quad \nabla.\mathbf{u} = 0. \end{aligned}\right\} \tag{1}$$

These lead to the equations

$$\left.\begin{aligned} \partial\mathbf{B}/\partial t+\mathbf{u}.\nabla\mathbf{B} &= \mathbf{B}.\nabla\mathbf{u}+\lambda\nabla^2\mathbf{B}, \\ \partial\mathbf{u}/\partial t+\mathbf{u}.\nabla\mathbf{u} &= \mathbf{B}.\nabla\mathbf{B}/\mu\rho-\nabla P/\rho+\nu\nabla^2\mathbf{u}, \end{aligned}\right\} \tag{2}$$

where
$$P = p+\mathbf{B}.\mathbf{B}/2\mu.$$

Consider now the case of a small disturbance in a fluid which is otherwise stationary in a uniform fixed magnetic field in, say, the z-direction. Then the total magnetic field can be expressed as $\mathbf{B} = (\mu\rho)^{\frac{1}{2}}(\mathbf{h}_0+\mathbf{h})$, where the factor $(\mu\rho)^{\frac{1}{2}}$ is introduced to give $\mathbf{h}_0$ and $\mathbf{h}$ the dimensions of velocity, $\mathbf{h}_0$ is a fixed vector in the z-direction of magnitude h_0, and $\mathbf{h}$ and $\mathbf{u} \ll \mathbf{h}_0$. If terms involving squares or products of $\mathbf{h}$ and $\mathbf{u}$ are neglected, equations (2) become

$$\left.\begin{aligned}(\partial/\partial t-\lambda\nabla^2)\,\mathbf{h} &= h_0(\partial\mathbf{u}/\partial z),\\ (\partial/\partial t-\nu\nabla^2)\,\mathbf{u} &= h_0(\partial\mathbf{h}/\partial z)-\nabla P/\rho.\end{aligned}\right\} \tag{3}$$

Outside the region of disturbance P is constant; inside it taking the divergence of these equations gives $\nabla^2 P = 0$. But a solution of Laplace's equation which is constant outside a bounded region is constant inside it also. Therefore P is constant throughout the fluid. With $\nabla P/\rho$ thus eliminated, equations (3) are the equations for Alfvén waves. They indicate that when λ and ν are small enough waves will travel at a speed h_0 from a point of disturbance either way along the stationary magnetic field.

Suppose now that the fluid is enclosed in a container with cylindrical symmetry about the z-axis. Then one can try to excite a mode in which the disturbances are purely circumferential. This is the simplest possible mode in a fluid which does not extend to infinity in some direction. Expressed in terms of cylindrical co-ordinates z, r, θ, the total magnetic and velocity fields will be of the form

$$\mathbf{B} = (\mu\rho)^{\frac{1}{2}}\{h_0, 0, h(z,r)\} \quad \text{and} \quad \mathbf{u} = \{0, 0, u(z,r)\}.$$

Equations (3) now give

$$\left.\begin{aligned}\left\{\frac{\partial}{\partial t}-\lambda\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\right\}h &= h_0\frac{\partial u}{\partial z},\\ \left\{\frac{\partial}{\partial t}-\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\right\}u &= h_0\frac{\partial h}{\partial z}.\end{aligned}\right\} \tag{4}$$

If the excitation is steady, standing waves will be produced. For these it is possible to put

$$h/h_0 = He^{i\omega t}, \quad u/h_0 = Ue^{i\omega t}.$$

To make possible the occurrence of resonance the depth of the container in the z-direction should be everywhere the same, so that the ratio of depth to wavelength is the same. The container should therefore be a cylinder or a torus of rectangular section. Henceforth the case of a torus will be treated: the case of a cylinder can be regarded as a special case for which the inner wall is at $r = 0$. It is convenient to take the origin at the centre of a container of depth $2d$, and to introduce the dimensionless quantities,

$$Z = z/d, \quad R = r/d, \quad T = h_0t/d, \quad \Omega = \omega d/h_0, \quad \Lambda = \lambda/h_0 d, \quad \mathrm{N} = \nu/h_0 d.$$

The walls of the container will then be at $Z = \pm 1$, $R = R_1$ and $R = R_2$ (figure 1), and equations (4) rewritten in terms of dimensionless quantities become

$$\left.\begin{aligned}\left\{i\Omega-\Lambda\left(\frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right)\right\}H &= \frac{\partial U}{\partial Z},\\ \left\{i\Omega-\mathrm{N}\left(\frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right)\right\}U &= \frac{\partial H}{\partial Z}.\end{aligned}\right\} \tag{5}$$

Λ is the reciprocal of the number introduced by Lundquist (Lundquist 1949) as a measure of the damping to be expected.

To excite standing waves it is necessary to establish an oscillating circumferential component of either magnetic field or velocity on a surface crossed by the stationary magnetic field. A convenient method of exciting waves magnetically in a torus is to supply alternating current to a winding around the torus. Then if the current in the winding approximates to a current sheet the boundary condition can be represented as

$$H = A/R, \quad U = 0 \quad \text{at} \quad Z = \pm 1, \quad R = R_1, \quad R = R_2. \tag{6}$$

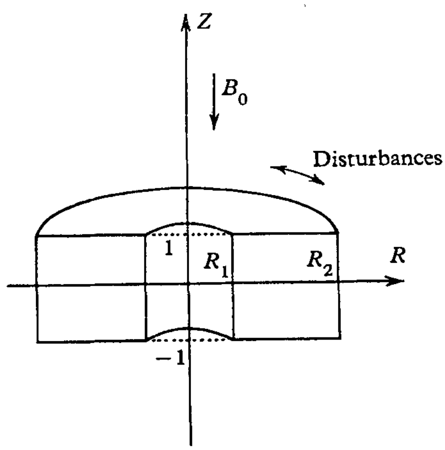

FIGURE 1. Co-ordinate system used.

The corresponding method of exciting waves mechanically is to oscillate the complete container. Then if the walls are non-conducting the boundary condition can be represented as

$$H = 0, \quad U = AR \quad \text{at} \quad Z = \pm 1, \quad R = R_1, \quad R = R_2. \tag{6'}$$

Since there is no longer any need to allow for a current circuit the container need not be a torus but could be a cylinder. Other methods of excitation, both magnetic and mechanical, could be devised. For example, current could be passed radially through the fluid between a central and a circumferential electrode; or just one face at $Z = +1$, say, could be oscillated. However, conditions (6) and (6′) will be taken as typical of magnetic and mechanical excitation.

In either case the solution can be obtained with the aid of expansions in the set of eigenfunctions

$$\mathscr{E}(n, R) = J_1(nR)\,Y_1(nR_1) - Y_1(nR)\,J_1(nR_1),$$

satisfying

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\right)\mathscr{E} = -n^2\mathscr{E},$$

where n takes the values $n_1, n_2, \ldots$, such that

$$\mathscr{E}(n, R_1) = \mathscr{E}(n, R_2) = 0.$$

In fact, where $E = \frac{1}{2}(R_2 - R_1)$ is the ratio of width to depth of the section,

$$\epsilon < (2E/\pi)\, n_\epsilon < \epsilon + \tfrac{1}{4},$$

$(2E/\pi)\, n_\epsilon$ increasing from ϵ towards $\epsilon+\frac{1}{4}$ as R_1/R_2 decreases from 1 to 0. Consider the case of magnetic excitation. An expansion in the eigenfunctions $\mathscr{E}(n,R)$ can only represent a function which vanishes at $R = R_1$ and $R = R_2$. One must therefore put

$$H = K + G,$$

where K is any suitably smooth function which satisfies the boundary condition (6) on $R = R_1$ and $R = R_2$, and find the expansions of G and U. According to condition (6) it suffices to take $K = A/R$. Then K represents the field that would be produced in free space, and

$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right)K = \frac{\partial K}{\partial Z} = 0,$$

so that equations (5) give

$$\left\{i\Omega - \Lambda\left(\frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right)\right\}G = \frac{\partial U}{\partial Z} - i\Omega K,$$

$$\left\{i\Omega - \mathrm{N}\left(\frac{\partial^2}{\partial Z^2}+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\right)\right\}U = \frac{\partial G}{\partial Z}.$$

Now expand K, G and U in the form

$$X(Z,R) = \sum_{\epsilon=1}^{\infty} \bar{X}(n_\epsilon, Z)\,\mathscr{E}(n_\epsilon, R).$$

Then all conditions are satisfied if for each value $n_1, n_2, \ldots$,

$$\left(i\Omega + \Lambda n^2 - \Lambda\frac{\partial^2}{\partial Z^2}\right)\bar{G} = \frac{\partial \bar{U}}{\partial Z} - i\Omega\bar{K},$$

$$\left(i\Omega + \mathrm{N}n^2 - \mathrm{N}\frac{\partial^2}{\partial Z^2}\right)\bar{U} = \frac{\partial \bar{G}}{\partial Z},$$

$$\bar{G} = \bar{U} = 0 \quad \text{when} \quad Z = \pm 1,$$

and these equations determine $\bar{G}(n,Z)$ and $\bar{U}(n,Z)$ in terms of $\bar{K}(n)$. In fact

$$\bar{K}(n) = A\pi\frac{(R_1/R_2)\,L(n)-1}{L(n)^2-1},$$

where
$$L(n) = J_1(nR_1)/J_1(nR_2) = Y_1(nR_1)/Y_1(nR_2),$$

and
$$\left.\begin{aligned}\bar{G} &= a\bar{K}\left(\frac{(c_1\cos k_1 Z/\sin k_1)-(c_2\cos k_2 Z/\sin k_2)}{c_1\cot k_1 - c_2\cot k_2} - 1\right),\\ \bar{U} &= a\bar{K}\left(\frac{(\sin k_1 Z/\sin k_1)-(\sin k_2 Z/\sin k_2)}{c_1\cot k_1 - c_2\cot k_2}\right),\end{aligned}\right\} \tag{7}$$

where
$$a = i\Omega/(i\Omega + \Lambda n^2),$$

$$k_1^2 = -\alpha + (\alpha^2-\beta)^{\frac{1}{2}}, \quad k_2^2 = -\alpha - (\alpha^2-\beta)^{\frac{1}{2}},$$

$$\alpha = \{1 + \Lambda(i\Omega + \mathrm{N}n^2) + \mathrm{N}(i\Omega + \Lambda n^2)\}/2\Lambda N,$$

$$\beta = (i\Omega + \Lambda n^2)(i\Omega + \mathrm{N}n^2)/\Lambda\mathrm{N},$$

$$c_1^2 = -\frac{i\Omega + \Lambda(n^2+k_1^2)}{i\Omega + \mathrm{N}(n^2+k_1^2)}, \quad c_2^2 = -\frac{i\Omega + \Lambda(n^2+k_2^2)}{i\Omega + \mathrm{N}(n^2+k_2^2)}.$$

The solution for mechanical excitation is obtained by the same method. In this case one must put

$$U = K + V,$$

where now K is any suitably smooth function satisfying condition (6′). Taking

$$K = AR,$$

again
$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\right) K = \frac{\partial K}{\partial Z} = 0.$$

The equations, and consequently their solution, are then the same as before with G and U replaced by V and H, Λ and N exchanged, and new values $\bar{K}(n)$.

To design an experiment one must obtain an idea of the overall form of these solutions. Take the case of magnetic excitation. As $n \to \infty$, $\bar{G}$ and $\bar{U} \to 0$. An indication is given by the earlier terms for which n is small. Considering these only, suppose that Λ and N are so small that

$$\Lambda\Omega \quad \text{and} \quad \mathrm{N}\Omega \ll 1, \quad \Lambda n^2/\Omega \quad \text{and} \quad \mathrm{N}n^2/\Omega \ll 1.$$

In this case
$$|a| \doteq 1, \quad \arg a \doteq \Lambda n^2/\Omega.$$

Also
$$|\alpha| \sim 1/\Lambda\mathrm{N}, \quad |\beta| \sim \Omega^2/\Lambda\mathrm{N}, \quad |\alpha|^2 \gg |\beta|,$$

$$k_1^2 \doteq -\beta/2\alpha, \quad k_2^2 \doteq -2\alpha,$$

whence
$$\begin{aligned}
|k_1| &\doteq \Omega, & \arg k_1 &\doteq \tfrac{1}{2}(\Lambda+\mathrm{N})(\Omega+n^2/\Omega),\\
|k_2| &\doteq 1/(\Lambda\mathrm{N})^{\frac{1}{4}}, & \arg k_2 - \tfrac{1}{2}\pi &\doteq -\tfrac{1}{2}(\Lambda+\mathrm{N})\,\Omega,\\
|c_1| &\doteq 1, & \arg c_1 + \tfrac{1}{2}\pi &\doteq \tfrac{1}{2}(\Lambda-\mathrm{N})\,\Omega,\\
|c_2| &\doteq (\mathrm{N}/\Lambda)^{\frac{1}{2}} = (\nu/\lambda)^{\frac{1}{2}}, & \arg c_2 + \tfrac{1}{2}\pi &\doteq -\tfrac{1}{2}(\Lambda-\mathrm{N})\,\Omega.
\end{aligned}$$

Since k_1 is mainly real, and k_2 is mainly imaginary, only $\cos k_1 Z$ and $\sin k_1 Z$ represent waves penetrating to the centre: $\cos k_2 Z$ and $\sin k_2 Z$ represent boundary layers dying away rapidly from the end walls at $Z = \pm 1$. For large imaginary k_2, $\cot k_2 \doteq -i$. On the central plane $Z = 0$, where the boundary-layer part is negligible, equations (7) thus give

$$\bar{G} \doteq \bar{K}\left(\frac{c_1}{c_1 \cos k_1 + ic_2 \sin k_1} - 1\right), \quad \bar{U} = 0.$$

If $\nu < \lambda$, so that $|c_2| < |c_1|$, $\bar{G}(n, Z)$ will be at a maximum for each value n, and consequently there will be resonances at which H is at maximum, when

$$\Omega = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \ldots,$$

$$\cos k_1 \doteq \pm i\tfrac{1}{2}(\Lambda+\mathrm{N})(\Omega^2+n^2), \quad \sin k_1 \doteq \pm 1.$$

Then on the central plane

$$\frac{\bar{G}}{\bar{K}} + 1 \doteq \pm \frac{1}{\frac{1}{2}(\Lambda+\mathrm{N})(\Omega^2+n^2)+(\nu/\lambda)^{\frac{1}{2}}}. \tag{8}$$

$|H| \gg K$ near the centre of the section if, at least for the first eigenvalue n_1, $|\bar{G}| \gg \bar{K}$. This is the case if both

$$\nu \ll \lambda \quad \text{and} \quad (\Lambda+\mathrm{N})(\Omega^2+n_1^2) \ll 1. \tag{9, 10}$$

The performance will be limited by whichever is larger of $(\nu/\lambda)^{\frac{1}{2}}$ and

$$\tfrac{1}{2}(\Lambda + \mathrm{N})(\Omega^2 + n_1^2).$$

Condition (9) is the condition that the wave part of the solution shall be dominant over the boundary-layer part. If $\nu > \lambda$ the wave part will be swamped by the boundary-layer part and H will not exceed K even when Λ and $\mathrm{N} \to 0$. Condition (10) is the condition that there will be a small amount of damping. When damping is the factor limiting the performance, the geometry should be designed so that n_1 does not appreciably exceed Ω: otherwise $|H|$ will be much smaller than the limit determined by $(\Lambda + \mathrm{N})\,\Omega$. Now n_1 is close to $\pi/2E$. There will be a penalty at the resonance for which $\Omega = \frac{1}{2}\delta\pi$, therefore, if the ratio E of width to depth of the section is much less than $1/\delta$. This ratio is extremely important. The ratio R_1/R_2, on the other hand, affects only the small difference between n_1 and $\pi/2E$, and is relatively unimportant.

Equation (8) also gives an indication of the spatial variation of H at resonance. Associated with the variation of H with Z there will be radial electric currents. These will be blocked by the side walls at $R = R_1$, $R = R_2$, and forced to bend parallel to the axis. In the absence of longitudinal currents produced in this way H would vary radially as $1/R$. Now if $\bar{G}(n, Z_0)/\bar{K}(n)$ is nearly independent of n as far as the ϵth term, H will approach a multiple of K on the plane $Z = Z_0$, and will therefore vary approximately as $1/R$, in the region which is more than a distance about $1/n_\epsilon$ times the half depth from each side wall. Within this distance of each side wall there will be a boundary region in which longitudinal currents will be important and H will vary rapidly with R. According to equation (8), at resonance $\bar{G}/\bar{K}$ is nearly independent of n when $n < \Omega$, so on the central plane the boundary regions then extend a distance about $1/\Omega$ times the half depth from each side wall. At the resonance for which $\Omega = \frac{1}{2}\pi\delta$ the whole section will be within the boundary regions if E is much less than δ^{-1}, and it is then that there is a loss of performance. Off resonance by a sufficient amount τ, still small, $\cos k_1 \doteq \pm\tau \pm i\frac{1}{2}(\Lambda + \mathrm{N})(\Omega^2 + n^2)$, $\sin k_1 \doteq \pm 1$, and $\bar{G}/\bar{K}$ is nearly independent of n until a higher value of n at which $(\Lambda + \mathrm{N})(\Omega^2 + n^2) \sim \tau$. The boundary regions will therefore be narrower. There is thus an expansion of these regions at resonance. Consequently the resistive losses associated with the longitudinal currents are smaller and the damping is less than it might have been.

These effects are illustrated by figures 2 and 3, in which the results of numerical calculations are presented for magnetically excited waves when $\Lambda = 0{\cdot}1$, $\mathrm{N} = 0$, so that $\nu/\lambda = 0$, and damping is the limiting factor. In all these calculations a computer was used to sum the first 20 non-vanishing terms of the series. Figure 2 shows the effect of changing R_1/R_2 when E is held fixed equal to 1, with a constant inner boundary value $H = 1$, and a varying outer boundary value $H = R_1/R_2$. It is convenient to take as a criterion of performance the ratio $|H|/K$ of the field produced in the fluid to the field produced in free space with the same excitation. The point at which $|H|$ is greatest is displaced inwards as R_1/R_2 decreases, but $|H|/K$ is not much affected: at the centre point of the section $|H|/K$ decreases from $4{\cdot}9$ to $4{\cdot}6$ as R_1/R_2 decreases from 1 to $\frac{1}{7}$. Figure 3 shows the effect of changing E in the limiting case when $R_1/R_2 = 1$ and the torus is replaced by an infinite straight channel. The importance of width is obvious. Lehnert's experiment (Lehnert 1954) suffered from too small a width.

If the first term $\bar{G}(n_1, Z)$ of the expansion is taken as a sufficient indication of H, it is possible to determine approximately the ratio E for which the performance at the first resonance will be at a maximum with a given volume of fluid. Λ and

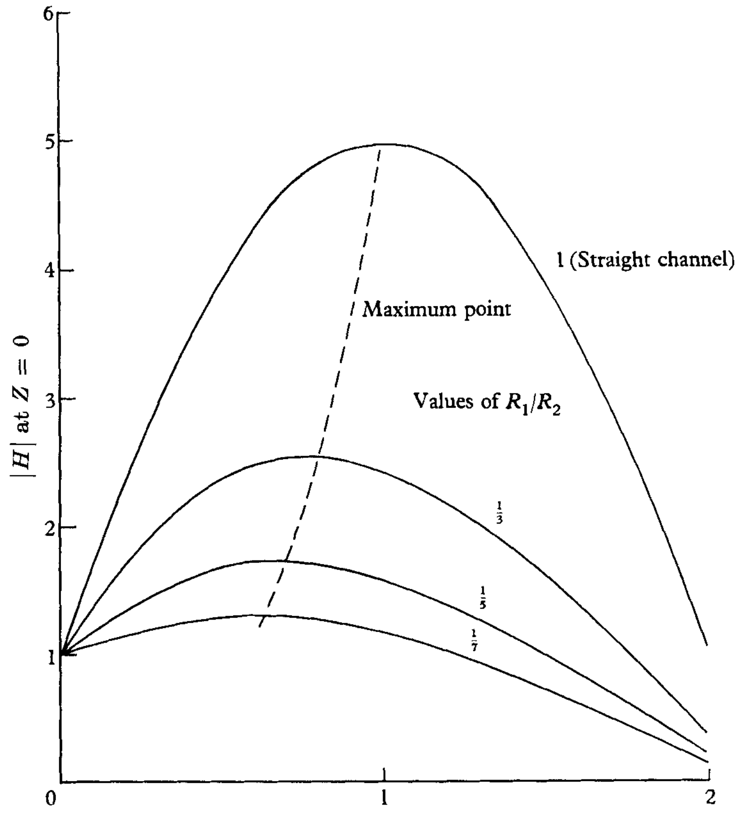

FIGURE 2. The effect of the ratio R_1/R_2: profiles of H across the central plane $Z = 0$ at the first resonance for different values of R_1/R_2 when $E = 1$. Inner boundary value $H = 1$. $\Lambda = 0{\cdot}1$. $\mathrm{N} = 0$.

N vary as $1/d$, where d is the half depth. The volume of fluid in a torus with a given ratio of inner to outer radius varies as d^3E^2, so, if this volume is fixed, Λ and N vary as $E^{\frac{2}{3}}$. Also n_1 is close to $\pi/2E$. At the first resonance, therefore, for which $\Omega = \frac{1}{2}\pi$, if $(\nu/\lambda)^{\frac{1}{2}}$ is small enough to be neglected, $\bar{G}(n_1, 0)/\bar{K}(n_1) + 1$ varies approximately as $1/(E^{\frac{2}{3}} + E^{-\frac{4}{3}})$ and is greatest when E is about $\sqrt{2}$.

If the excitation is mechanical the symmetry of the equations indicates that condition (9) must be replaced by

$$\lambda \ll \nu. \qquad (9')$$

Since $\nu \ll \lambda$ for real liquids, condition (9) indicates that magnetic excitation should be used.

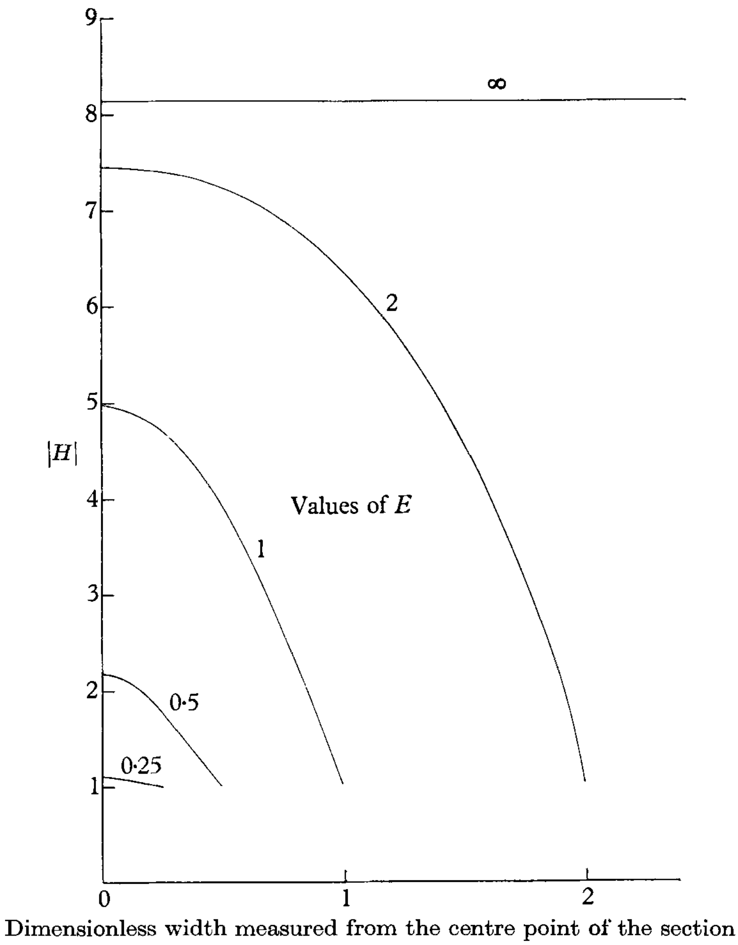

FIGURE 3. The effect of width in the case of a channel: profiles of H across the central plane $Z = 0$ at the first resonance for different values of E when $R_1/R_2 = 1$. Boundary value $H = 1$. $\Lambda = 0{\cdot}1$. $\mathrm{N} = 0$.

3. Apparatus

The most readily available liquid conductors are mercury, sodium–potassium alloy, and heated sodium. If a strong first resonance is to be produced, an acceptable maximum value of $\Lambda + \mathrm{N}$ is about 0·1. Using these liquids in a magnetic field of 1·0 Weber/m^2, this figure is obtained with half depths of 196, 19·5 and 5·6 cm, respectively. Sodium was therefore chosen.

Mechanical excitation being ruled out because for sodium $\nu \ll \lambda$, the simplest possible method of magnetic excitation was adopted.† The sodium was contained inside a torus lying in a strong magnetic field parallel to its axis, and waves were generated by means of alternating current supplied to a coil around the torus. An advantage of this arrangement was that the sodium could be sealed once and for all inside the container.

† The method of excitation used by Lehnert (Lehnert 1954) depended on the electric conductivity of the disk, and was essentially magnetic.

The container (figures 4 and 5) was designed to fill a space 20 cm deep and 60 cm in diameter. It was made by welding together sections of stainless-steel sheet. This material was selected for its strength, resistance to sodium, non-magnetic behaviour, and relatively low electrical conductivity. Nevertheless, the conductivity was not so low that currents induced in the walls could be ignored, and in order to reduce this complication sheet only 1·2 mm thick was used. The container was quite flexible and needed to be well supported. To ensure good welds at the rims its upper and lower faces were dished. It was

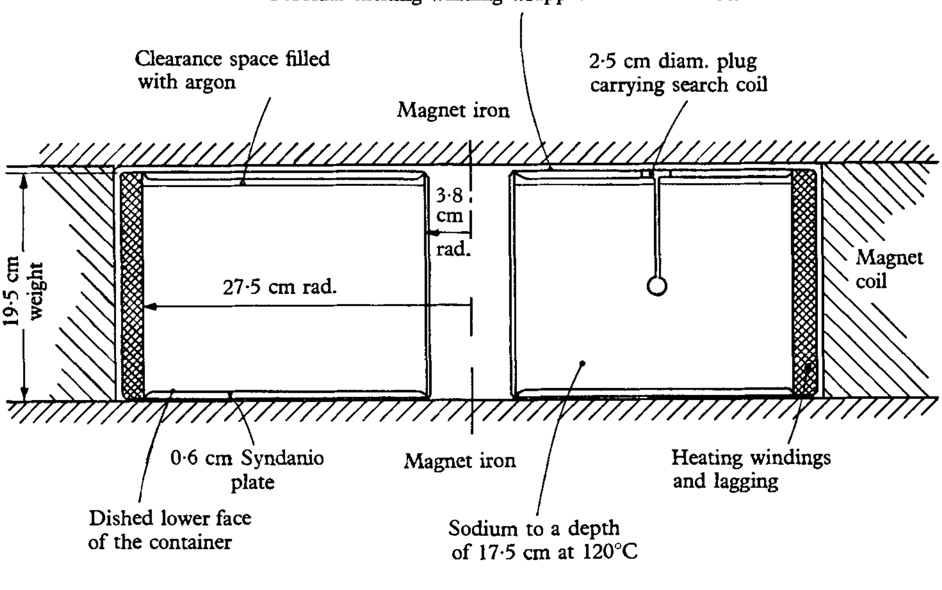

FIGURE 4. The container.

filled under an atmosphere of argon, and a clearance space 0·6 cm deep was left at the top to help allow for expansion and contraction of the sodium. It was sealed at a temperature of 120° C. Experiments were conducted at the same temperature to prevent the container being deformed as a result of a pressure difference across its walls. At this temperature the sodium occupied a section 17·5 cm deep with an outer radius of 27·4 cm and an inner radius of 3·9 cm. The sodium was heated by three resistance elements wound around the outer circumference of the container. These were connected to separate phases of the a.c. supply so that they produced no magnetic effect. To reduce the loss of heat Syndanio plates were placed in the dished faces, and asbestos cord was wound outside the heating elements. The temperature was measured by a thermocouple at the inner circumference.

A small search coil 1·9 cm in diameter enclosed in a stainless-steel case was set at the centre of the fluid section facing around the circumference, supported by a stalk from the sealing cap. In addition a large search coil was wound around the whole section outside the container walls, but under the heating elements, enclosing an area 19·6 cm deep, with an outer radius of 27·6 cm and an inner radius of 3·7 cm. The e.m.f. induced in these coils thus gave respectively an indication of the field at the centre and of the average field through the whole section. No provision was made for measuring the fluid velocity. The exciting coil, which was wound outside everything else, was designed to carry a current which would approximate as closely as possible to a current sheet. It consisted of 20 sectors of copper sheet insulated with fibreglass and connected in series.

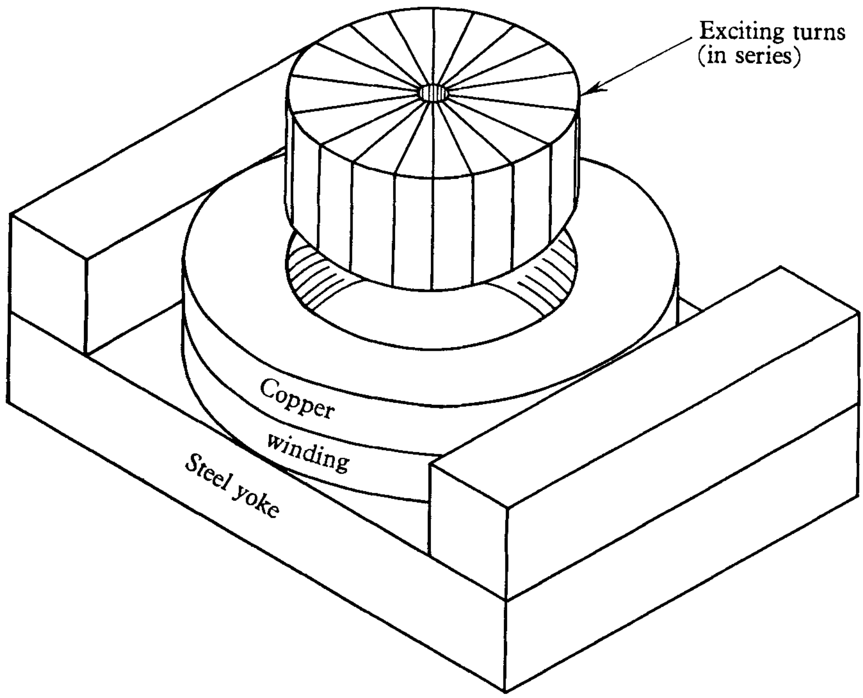

FIGURE 5. Sodium container before being lowered into the magnet (top of magnet yoke removed).

The complete container was lowered on fibreglass tapes into the cavity of a specially constructed magnet (figure 5) which provided the fixed axial component of the field along which waves could travel. The magnet consisted simply of a coil sandwiched between two flat steel slabs separated by spacer pieces which also provided the return path for the field. With slabs of sufficiently high permeability this arrangement is equivalent to a slice out of a long solenoid, and it was found to give a field uniform within the tolerances of manufacture, $\pm 3\,\%$. To expose the cavity the complete lid of the magnet was removed by means of a crane in the laboratory roof. The magnet coil was not cooled, and its temperature rose sufficiently rapidly for it to be difficult to maintain the current constant when the field was raised above 0·7 Weber/m^2, although 1·0 Weber/m^2 could be produced for a short period. The complete magnet weighed 8·4 ton.

Current to generate waves was fed to the exciting coil from a 100 W variable-frequency oscillator through a matching transformer. A measured resistance was placed in series with the exciting coil, and the p.d. across this used to indicate the current. This signal was compared in magnitude and phase with the signal from one or other of the search coils by means of a Tektronix twin-beam oscilloscope. Signals from the small central search coil had first to be passed through a pre-amplifier. The combination of the pre-amplifier and the oscilloscope amplifier could be calibrated by means of a standard square wave from a source inside the oscilloscope, accurate to $\pm 3\,\%$.

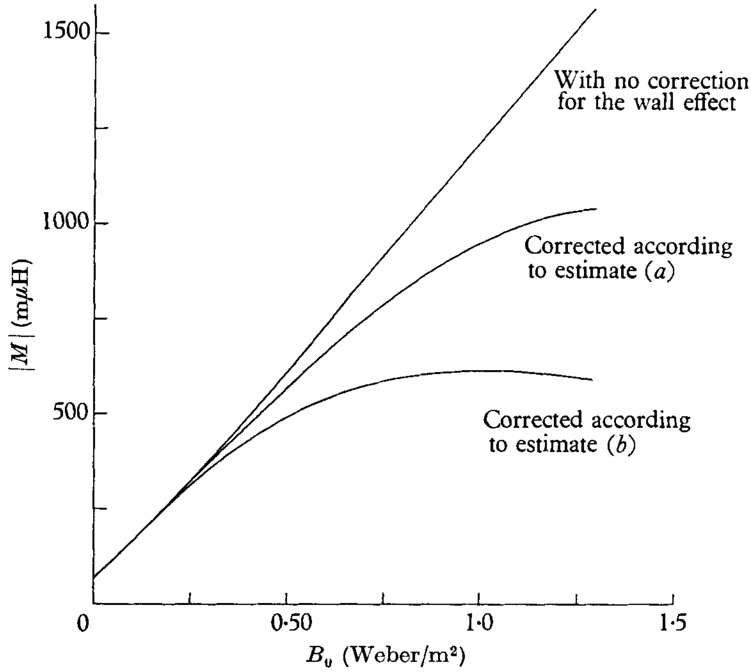

FIGURE 6. M at the first resonance for the central search coil calculated with and without corrections for the wall effect.

4. Calculation of performance

Using complex numbers to represent the magnitude and phase of alternating quantities, we suppose that an e.m.f. V is induced in one or other search coil when the system is excited by the application of a current I. Then a convenient way of measuring the performance is in terms of the coupling M between the search coil and the exciting coil, defined by

$$V = i\omega MI.$$

In free space M is the mutual inductance and is real, but in the presence of sodium M contains a phase factor. The main difficulty in estimating M is to allow for currents induced in the container walls. It is possible to make an estimate assuming

(*a*) that the walls are completely insulated from the sodium, or

(*b*) that all the walls of the container are in contact with the sodium, with no contact resistance.

Estimate (*a*). Where the walls are completely insulated from the sodium, we let Φ be the flux when unit total exciting current links the torus. Φ can be calculated by integrating term by term the expansion for H given by equations (7), and adding a contribution from the clearance space above the sodium. Then, if I_0 is the total current linking the torus applied in the exciting coil, I the total effective exciting current linking the torus after subtraction of the wall current, and R the resistance of the wall circuit,

$$I = I_0 - (i\omega\Phi I/R), \quad \text{whence} \quad I = I_0/\{(1+i\omega\Phi)/R\}.$$

Using this result it is possible to calculate H for a given applied current I_0 and so to obtain an estimate of M for each search coil.

Estimate (*b*). Where all the walls of the container are in contact with the sodium, with no contact resistance, the sides (parallel to the axis) and the faces (perpendicular to the axis) may be expected to act almost independently, because of the free flow of current into and out of the walls. Conducting sides improve the performance by absorbing part of the transverse current associated with the waves, and in the limit of perfectly conducting sides the situation becomes the same as for an infinitely wide region. Conducting faces inhibit the performance by carrying currents acting against the exciting current, and in the limit of perfectly conducting faces the situation becomes a pure skin-effect at the sides.

With conducting sides only, the equations can be solved, neglecting viscosity, with separated variables by means of an expansion of eigenfunctions depending on Z. On the assumption of completely independent action of the sides and faces, an estimate was obtained by this procedure that in the régime of the experiments the effect of the sides should be less than 2 %. When this contribution is neglected, it becomes possible to solve the equations for a container with conducting faces by the method of §2. Where F is the ratio of the wall thickness to the half depth of the fluid, Λ_0 the dimensionless electric diffusivity of the wall material and

$$a_0^2 = i\Omega/(i\Omega+\Lambda_0 n^2), \quad k_0^2 = (i\Omega/\Lambda_0)+n^2,$$
$$s_1 = \Lambda k_1/\Lambda_0 k_0, \quad s_2 = \Lambda k_2/\Lambda_0 k_0,$$

equations (7) have to be modified to

$$\bar{G} = a\bar{K}\left(\frac{\{1+(a_0/a)(\operatorname{sech} k_0 F - 1)\}(c_1 \cos k_1 Z/\sin k_1 - c_2 \cos k_2 Z/\sin k_2)}{c_1(\cot k_1 - s_1 \tanh k_0 F) - c_2(\cot k_2 - s_2 \tanh k_0 F)} - 1\right).$$

Using this formula it is again possible to obtain an estimate of M.

Since the top face of the container was separated from the sodium by the clearance space while the other walls were in contact with it, neither estimate can be exact. Each is incorrect not only in the boundary conditions assumed for H, but also in the condition assumed for U at the free surface, which should be $\partial U/\partial Z = 0$. However, since $\nu \ll \lambda$, the error introduced by this second incorrect assumption should be small, and one can expect that the actual behaviour will be somewhere in between what is predicted by estimates (*a*) and (*b*), which should represent limits for the possible performance.

According to either estimate the wall effect proves to be important in the régime of the experiments only at the first resonance, and not at other frequencies.

The reduction in performance at the first resonance increases quite rapidly as the fixed component of the field is increased because of the increase both in the resonant magnification of the alternating component of the field, and in the

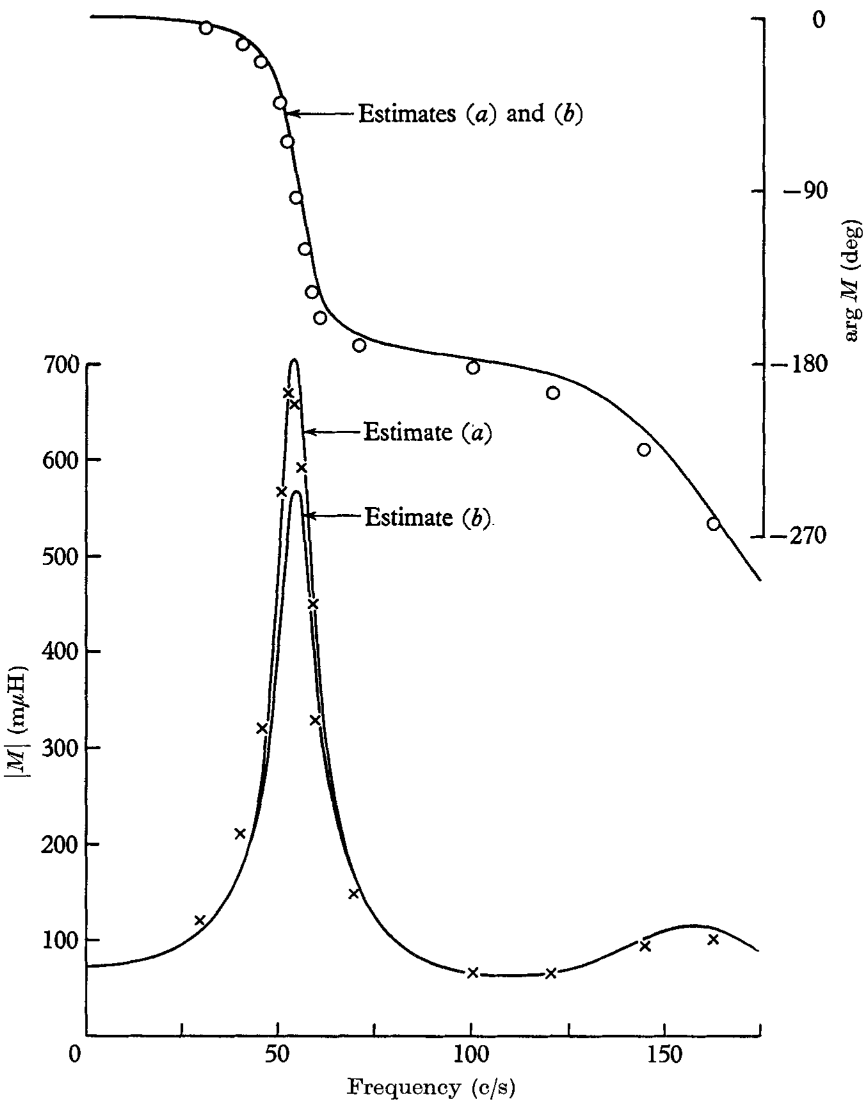

FIGURE 7. Comparison of measurements with calculated curves for the central search coil. Field = 0·650 Weber/m^2; Λ+N = 0·0495; ×, ○, measurements; ———, calculated curves.

resonant frequency. Figure 6 shows the reduction in $|M|$ at the first resonance according to the two estimates for the central search coil. Because of the wall effect, when the fixed component of the field exceeds 1·0 Weber/m^2, the performance ceases to increase and finally declines.

In all the calculations the properties of sodium at 120 °C were assumed to be

$$\lambda = 0{\cdot}825 \times 10^{-1}\,\mathrm{m^2/s}, \quad \nu = 0{\cdot}677 \times 10^{-6}\,\mathrm{m^2/s}, \quad \rho = 0{\cdot}93 \times 10^{3}\,\mathrm{kg/m^2},$$

and of stainless steel $\lambda = 0{\cdot}565\,\mathrm{m^2/s}$.

In every case the first 20 terms of the expansion were summed. For the central search coil M was calculated simply by multiplying H at the centre of the section by the area of the coil. For the full-section coil it was necessary to add to

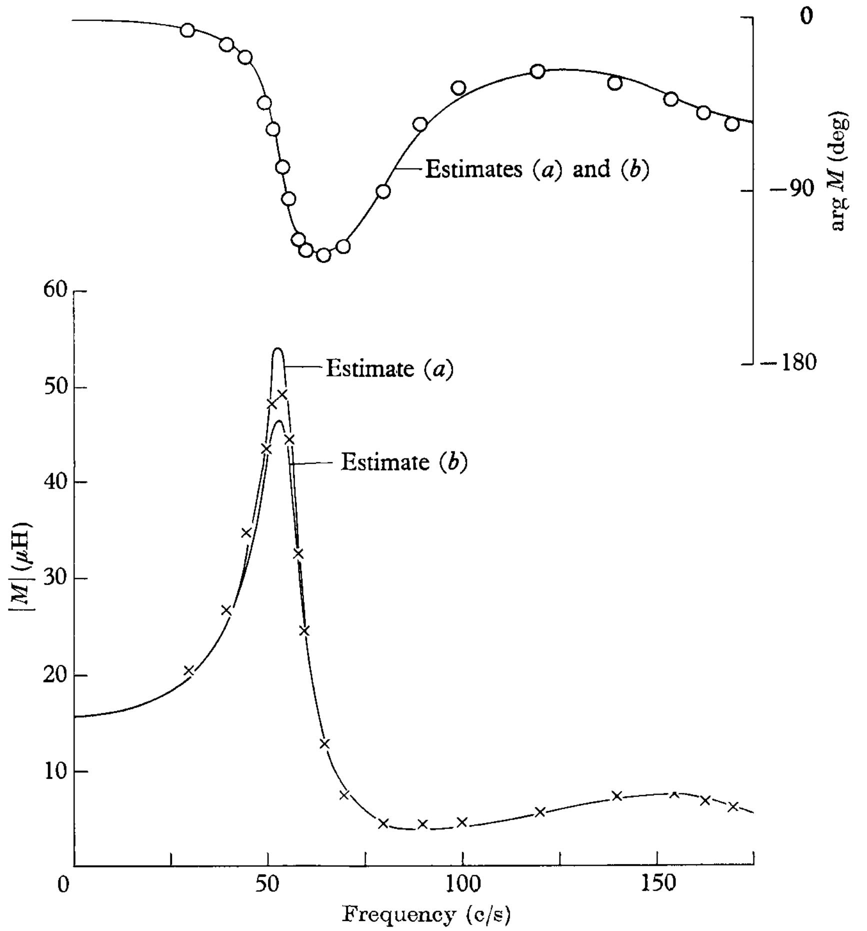

FIGURE 8. Comparison of measurements with calculated curves for the full section search coil. Field = 0·650 Weber/m²; $\Lambda + N = 0{\cdot}0495$; ×, ○, measurements; ———, calculated curves.

the contribution from the sodium a contribution from the walls and the space inside and outside the walls between the sodium and the search coil. This contribution was estimated to be $2{\cdot}0\,\mu$H.

5. Results

In each test the frequency was varied while the axial magnetic field was held constant within the error of observation, giving a constant value of $\Lambda + N$. Enough current was applied in the exciting coil to create small disturbances only: the ratio of the alternating component to the fixed component of the field was never more than 0·007 anywhere in the section.

Results obtained using the central and the full-section search coils respectively are presented in figures 7 and 8. Each figure shows measured values of $|M|$ and $\arg M$ at different measured values of the frequency in a field measured to be 0·65 Weber/m^2, giving $\Lambda + \mathrm{N} = 0{\cdot}0495$. In each figure, the estimates (*a*) and (*b*) of the dependence of M on the frequency at this field strength are also presented for comparison. It can be seen that, in addition to a strong first resonance at the fundamental frequency, a weak second resonance was obtained at three times the fundamental frequency. In free space M would be 70 mμH for the central search coil, 15·6 μH for the full-section search coil. The results for the central search coil indicate that at the centre the alternating component of the field reached a peak magnitude 9·6 times the magnitude of the field that would have been generated in free space by the exciting current.

Account should be taken of possible errors of about $\pm 5\,\%$ in the measurements $|M|$ and $\arg M$, $\pm 2\,\%$ in the measurement of the frequency, and $\pm 5\,\%$ in the measurement of the vertical field. Account should also be taken of possible errors in the conductivity, density and dimensions assumed in the calculations. Of these the most important was likely to have been a possible error of about $\pm 2\,\%$ in the conductivity of the sodium, due to inexact control of the temperature, apart from any error due to impurities. In this light the agreement between the measurements and the calculations is satisfactory. The values of $|M|$ measured at the first resonance lie well between the estimates (*a*) and (*b*). The slightly high values of $|M|$ measured at the first resonance, and the low values measured at the second resonance, in the case of the central search coil, would be consistent with a contribution from the flux linking its leads. Since it was impossible to gain access to this coil once it had been encased, its leads were not twisted for fear of damaging the insulation.

The prediction that standing waves producing a strong resonance could be generated by this apparatus was thus well confirmed. Experiments to generate travelling waves are also being conducted.

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